

Normal transversality and uniform bounds

FRANCESC PLANAS-VILANOVA

Dept. Matemàtica Aplicada 1. ETSEIB-UPC. Diagonal 647, 08028 Barcelona. E-mail: planas@ma1.upc.es

1 Introduction

Let A be a commutative ring. A graded A -algebra $U = \bigoplus_{n \geq 0} U_n$ is a *standard* A -algebra if $U_0 = A$ and $U = A[U_1]$ is generated as an A -algebra by the elements of U_1 . A graded U -module $F = \bigoplus_{n \geq 0} F_n$ is a *standard* U -module if F is generated as an U -module by the elements of F_0 , that is, $F_n = U_n F_0$ for all $n \geq 0$. In particular, $F_n = U_1 F_{n-1}$ for all $n \geq 1$. Given I, J , two ideals of A , we consider the following standard algebras: the *Rees algebra* of I , $\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n t^n = A[It] \subset A[t]$, and the *multi-Rees algebra* of I and J , $\mathcal{R}(I, J) = \bigoplus_{n \geq 0} (\bigoplus_{p+q=n} I^p J^q u^p v^q) = A[Iu, Jv] \subset A[u, v]$. Consider the *associated graded ring* of I , $\mathcal{G}(I) = \mathcal{R}(I) \otimes A/I = \bigoplus_{n \geq 0} I^n / I^{n+1}$, and the *multi-associated graded ring* of I and J , $\mathcal{G}(I, J) = \mathcal{R}(I, J) \otimes A/(I + J) = \bigoplus_{n \geq 0} (\bigoplus_{p+q=n} I^p J^q / (I + J) I^p J^q)$. We can always consider the tensor product of two standard A -algebras $U = \bigoplus_{p \geq 0} U_p$ and $V = \bigoplus_{q \geq 0} V_q$ as an standard A -algebra with the natural grading $U \otimes V = \bigoplus_{n \geq 0} (\bigoplus_{p+q=n} U_p \otimes V_q)$. If M is an A -module, we have the standard modules: the *Rees module* of I with respect to M , $\mathcal{R}(I; M) = \bigoplus_{n \geq 0} I^n M t^n = M[It] \subset M[t]$ (a standard $\mathcal{R}(I)$ -module), and the *multi-Rees module* of I and J with respect to M , $\mathcal{R}(I, J; M) = \bigoplus_{n \geq 0} (\bigoplus_{p+q=n} I^p J^q M u^p v^q) = M[Iu, Jv] \subset M[u, v]$ (a standard $\mathcal{R}(I, J)$ -module). Consider the *associated graded module* of I with respect to M , $\mathcal{G}(I; M) = \mathcal{R}(I; M) \otimes A/I = \bigoplus_{n \geq 0} I^n M / I^{n+1} M$ (a standard $\mathcal{G}(I)$ -module), and the *multi-associated graded module* of I and J with respect to M , $\mathcal{G}(I, J; M) = \mathcal{R}(I, J; M) \otimes A/(I + J) = \bigoplus_{n \geq 0} (\bigoplus_{p+q=n} I^p J^q M / (I + J) I^p J^q M)$ (a standard $\mathcal{G}(I, J)$ -module). If U, V are two standard A -algebras and F is a standard U -module and G is a standard V -module, then $F \otimes G = \bigoplus_{n \geq 0} (\bigoplus_{p+q=n} F_p \otimes G_q)$ is a standard $U \otimes V$ -module.

Denote by $\pi : \mathcal{R}(I) \otimes \mathcal{R}(J; M) \rightarrow \mathcal{R}(I, J; M)$ and $\sigma : \mathcal{R}(I, J; M) \rightarrow \mathcal{R}(I + J; M)$ the natural surjective graded morphisms of standard $\mathcal{R}(I) \otimes \mathcal{R}(J)$ -modules. Let $\varphi : \mathcal{R}(I) \otimes \mathcal{R}(J; M) \rightarrow \mathcal{R}(I + J; M)$ be $\sigma \circ \pi$. Denote by $\bar{\pi} : \mathcal{G}(I) \otimes \mathcal{G}(J; M) \rightarrow \mathcal{G}(I, J; M)$ and $\bar{\sigma} : \mathcal{G}(I, J; M) \rightarrow \mathcal{G}(I + J; M)$ the tensor product of π and σ by $A/(I + J)$; these are two natural surjective graded morphisms of standard $\mathcal{G}(I) \otimes \mathcal{G}(J)$ -modules. Let $\bar{\varphi} : \mathcal{G}(I) \otimes \mathcal{G}(J; M) \rightarrow \mathcal{G}(I + J; M)$ be $\bar{\sigma} \circ \bar{\pi}$. The first purpose of this note is to prove the following theorem:

Theorem 1 *Let A be a noetherian ring, I, J two ideals of A and M a finitely generated A -module. The following two conditions are equivalent:*

- (i) $\bar{\varphi} : \mathcal{G}(I) \otimes \mathcal{G}(J; M) \rightarrow \mathcal{G}(I + J; M)$ is an isomorphism.
- (ii) $\text{Tor}_1(A/I^p, \mathcal{R}(J; M)) = 0$ and $\text{Tor}_1(A/I^p, \mathcal{G}(J; M)) = 0$ for all integers $p \geq 1$.

In particular, $\mathcal{G}(I) \otimes \mathcal{G}(J) \simeq \mathcal{G}(I + J)$ if and only if $\text{Tor}_1(A/I^p, A/J^q) = 0$ and $\text{Tor}_2(A/I^p, A/J^q) = 0$ for all integers $p, q \geq 1$.

The morphism $\overline{\varphi}$ has been studied by Hironaka [H], Grothendieck [G] and Hermann, Ikeda and Orbanz [HIO], among others, but assuming always A is normally flat along I (see 21.11 in [HIO]). We will see how Theorem 1 generalizes all this former work.

Let us now recall some definitions in order to state the second purpose of this note. If U is a standard A -algebra and F is a graded U -module, put $s(F) = \min\{r \geq 1 \mid F_n = 0 \text{ for all } n \geq r+1\}$, where $s(F)$ may possibly be infinite. If $U_+ = \bigoplus_{n>0} U_n$ and $r \geq 1$, the following three conditions are equivalent: F can be generated by elements of degree at most r ; $s(F/U_+F) \leq r$; and $F_n = U_1 F_{n-1}$ for all $n \geq r+1$. If $\varphi : G \rightarrow F$ is a surjective graded morphism of graded U -modules, we denote by $E(\varphi)$ the graded A -module $E(\varphi) = \ker \varphi / U_+ \ker \varphi = \ker \varphi_0 \oplus (\bigoplus_{n \geq 1} \ker \varphi_n / U_1 \ker \varphi_{n-1}) = \bigoplus_{n \geq 0} E(\varphi)_n$. If F is a standard U -module, take $\mathbf{S}(U_1)$ the symmetric algebra of U_1 , $\alpha : \mathbf{S}(U_1) \rightarrow U$ the surjective graded morphism of standard A -algebras induced by the identity on U_1 and $\gamma : \mathbf{S}(U_1) \otimes F_0 \xrightarrow{\alpha \otimes 1} U \otimes F_0 \rightarrow F$ the composition of $\alpha \otimes 1$ with the structural morphism. Since F is a standard U -module, γ is a surjective graded morphism of graded $\mathbf{S}(U_1)$ -modules. The *module of effective n -relations* of F is defined to be $E(F)_n = E(\gamma)_n = \ker \gamma_n / U_1 \ker \gamma_{n-1}$ (for $n = 0$, $E(F)_n = 0$). Put $E(F) = \bigoplus_{n \geq 1} E(F)_n = \bigoplus_{n \geq 1} E(\gamma)_n = E(\gamma) = \ker \gamma / \mathbf{S}_+(\mathbf{S}(U_1)) \ker \gamma$. The *relation type* of F is defined to be $\text{rt}(F) = s(E(F))$, that is, $\text{rt}(F)$ is the minimum positive integer $r \geq 1$ such that the effective n -relations are zero for all $n \geq r+1$. A *symmetric presentation* of a standard U -module F is a surjective graded morphism of standard V -modules $\varphi : G \rightarrow F$, with $\varphi : G = V \otimes M \xrightarrow{f \otimes h} U \otimes F_0 \rightarrow F$, where V is a symmetric A -algebra, $f : V \rightarrow U$ is a surjective graded morphism of standard A -algebras, $h : M \rightarrow F_0$ is an epimorphism of A -modules and $U \otimes F_0 \rightarrow F$ is the structural morphism. One can show (see [P₂]) that $E(F)_n = E(\varphi)_n$ for all $n \geq 2$ and $s(E(F)) = s(E(\varphi))$. Thus the module of effective n -relations and the relation type of a standard U -module are independent of the chosen symmetric presentation. Roughly speaking, the relation type of F is the largest degree of any minimal homogeneous system of generators of the submodule defining F as a quotient of a polynomial ring with coefficients in F_0 . For an ideal I of A and an A -module M , the module of effective n -relations and the relation type of I with respect to M are defined to be $E(I; M)_n = E(\mathcal{R}(I; M))_n$ and $\text{rt}(I; M) = \text{rt}(\mathcal{R}(I; M))$, respectively. Then:

Theorem 2 *Let A be a commutative ring, U and V two standard A -algebras, F a standard U -module and G a standard V -module. Then $U \otimes V$ is a standard A -algebra, $F \otimes G$ is a standard $U \otimes V$ -module and $\text{rt}(F \otimes G) \leq \max(\text{rt}(F), \text{rt}(G))$.*

As a consequence of Theorems 1 and 2, one deduces the existence of a uniform bound for the relation type of all maximal ideals of an excellent ring.

Theorem 3 *Let A be an excellent (or $J-2$) ring and let M be a finitely generated A -module. Then there exists an integer $s \geq 1$ such that, for all maximal ideals \mathfrak{m} of A , the relation type of \mathfrak{m} with respect to M satisfies $\text{rt}(\mathfrak{m}; M) \leq s$.*

In fact, Theorem 3 could also be deduced from the proof of Theorem 4 of Trivedi in [T]. Finally, and using Theorem 2 of [P₂], one can recover the following result of Duncan and O'Carroll.

Corollary 4 [DO] *Let A be an excellent (or $J-2$) ring and let $N \subseteq M$ be two finitely generated A -modules. Then there exists an integer $s \geq 1$ such that, for all integers $n \geq s$ and for all maximal ideals \mathfrak{m} of A , $\mathfrak{m}^n M \cap N = \mathfrak{m}^{n-s}(\mathfrak{m}^s M \cap N)$.*

2 Normal transversality

Lemma 2.1 *Let A be a commutative ring, I an ideal of A , U a standard A -algebra, F and G two standard U -modules and $\varphi : G \rightarrow F$ a surjective graded morphism of standard A -algebras. If $\overline{A} = A/I$, then $\overline{U} = U \otimes \overline{A}$ is a standard \overline{A} -algebra, $\overline{F} = F \otimes \overline{A}$ and $\overline{G} = G \otimes \overline{A}$ are two standard \overline{U} -modules and $\overline{\varphi} = \varphi \otimes 1_{\overline{A}} : \overline{G} \rightarrow \overline{F}$ is a surjective graded morphism of standard \overline{U} -modules. Moreover, $s(E(\overline{\varphi})) \leq s(E(\varphi))$.*

Proof. Consider the following commutative diagram of exact rows:

$$\begin{array}{ccccccc}
 & U_1 \otimes \ker \varphi_{n-1} & \longrightarrow & U_1 \otimes G_{n-1} & \xrightarrow{1 \otimes \varphi_{n-1}} & U_1 \otimes F_{n-1} & \longrightarrow 0 \\
 & \downarrow & & \downarrow \partial_n^G & & \downarrow \partial_n^F & \\
 0 & \longrightarrow & \ker \varphi_n & \longrightarrow & G_n & \xrightarrow{\varphi_n} & F_n \longrightarrow 0.
 \end{array}$$

By the snake lemma, $\ker \partial_n^G \rightarrow \ker \partial_n^F \rightarrow E(\varphi)_n \rightarrow 0$ is an exact sequence of A -modules. If we tensor this sequence by \overline{A} , then $(\ker \partial_n^G) \otimes \overline{A} \rightarrow (\ker \partial_n^F) \otimes \overline{A} \rightarrow E(\varphi)_n \otimes \overline{A} \rightarrow 0$ is an exact sequence of \overline{A} -modules. On the other hand, we have the following commutative diagram of exact rows:

$$\begin{array}{ccccccc}
 & \overline{U}_1 \otimes \ker \overline{\varphi}_{n-1} & \longrightarrow & \overline{U}_1 \otimes \overline{G}_{n-1} & \xrightarrow{1 \otimes \overline{\varphi}_{n-1}} & \overline{U}_1 \otimes \overline{F}_{n-1} & \longrightarrow 0 \\
 & \downarrow & & \downarrow \partial_n^{\overline{G}} & & \downarrow \partial_n^{\overline{F}} & \\
 0 & \longrightarrow & \ker \overline{\varphi}_n & \longrightarrow & \overline{G}_n & \xrightarrow{\overline{\varphi}_n} & \overline{F}_n \longrightarrow 0.
 \end{array}$$

By the snake lemma, $\ker \partial_n^{\overline{G}} \rightarrow \ker \partial_n^{\overline{F}} \rightarrow E(\overline{\varphi})_n \rightarrow 0$ is an exact sequence of \overline{A} -modules. In order to see the relationship between $\ker \partial_n^{\overline{F}}$ and $\ker \partial_n^{\overline{G}}$, tensor by \overline{A} the exact sequence of A -modules $0 \rightarrow \ker \partial_n^F \rightarrow U_1 \otimes F_{n-1} \xrightarrow{\partial_n^F} F_n \rightarrow 0$ and consider the commutative diagram of exact rows:

$$\begin{array}{ccccccc}
 & (\ker \partial_n^F) \otimes \overline{A} & \longrightarrow & (U_1 \otimes F_{n-1}) \otimes \overline{A} & \xrightarrow{\partial_n^F \otimes 1} & F_n \otimes \overline{A} & \longrightarrow 0 \\
 & & & \downarrow \simeq & & \downarrow \simeq & \\
 0 & \longrightarrow & \ker \partial_n^{\overline{F}} & \longrightarrow & \overline{U}_1 \otimes \overline{F}_{n-1} & \xrightarrow{\partial_n^{\overline{F}}} & \overline{F}_n \longrightarrow 0.
 \end{array}$$

It induces an epimorphism of \overline{A} -modules $(\ker \partial_n^F) \otimes \overline{A} \rightarrow \ker \partial_n^{\overline{F}}$. Analogously, there exists an epimorphism of \overline{A} -modules $(\ker \partial_n^G) \otimes \overline{A} \rightarrow \ker \partial_n^{\overline{G}}$. Both epimorphisms make commutative the following diagram of exact rows:

$$\begin{array}{ccccccc}
 & (\ker \partial_n^G) \otimes \overline{A} & \longrightarrow & (\ker \partial_n^F) \otimes \overline{A} & \longrightarrow & E(\varphi)_n \otimes \overline{A} & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & & \\
 & \ker \partial_n^{\overline{G}} & \longrightarrow & \ker \partial_n^{\overline{F}} & \longrightarrow & E(\overline{\varphi})_n & \longrightarrow 0,
 \end{array}$$

from where we deduce an epimorphism $E(\varphi)_n \otimes \overline{A} \rightarrow E(\overline{\varphi})_n$. In particular, $s(E(\overline{\varphi})) \leq s(E(\varphi))$. ■

Lemma 2.2 *Let A be a commutative ring, I, J two ideals of A and M an A -module. Consider $\sigma : \mathcal{R}(I, J; M) \rightarrow \mathcal{R}(I + J; M)$ and $\overline{\sigma} = \mathcal{G}(I, J; M) \rightarrow \mathcal{G}(I + J; M)$. Then*

- (a) $\ker(\sigma_1) \simeq IM \cap JM$.
- (b) $\ker(\bar{\sigma}_1) = 0$ if and only if $IM \cap JM \subset I(I+J)M \cap (I+J)JM$.
- (c) If $I^p M \cap J^q M = I^p J^q M$ for all integers $p, q \geq 1$, then $s(E(\sigma)) = 1$ and $\bar{\sigma}$ is an isomorphism.

Proof. Consider $0 \rightarrow IM \cap JM \xrightarrow{\rho} IM \oplus JM \xrightarrow{\sigma_1} (I+J)M \rightarrow 0$ where $\rho(a) = (a, -a)$ and $\sigma_1(a, b) = a + b$. Clearly it is an exact sequence of A -modules. Thus $\ker(\sigma_1) = \rho(IM \cap JM) \simeq IM \cap JM$. If we tensor this exact sequence by $A/(I+J)$ we get $(IM \cap JM) \otimes A/(I+J) \xrightarrow{\bar{\rho}} (IM \oplus JM) \otimes A/(I+J) \xrightarrow{\bar{\sigma}_1} (I+J)M/(I+J)^2 M \rightarrow 0$. Then

$$\ker(\bar{\sigma}_1) = \text{im } \bar{\rho} = \{(\bar{a}, -\bar{a}) \in IM/I(I+J)M \oplus JM/(I+J)JM \mid a \in IM \cap JM\}.$$

Hence $\ker(\bar{\sigma}_1) = 0$ if and only if $IM \cap JM \subset I(I+J)M \cap (I+J)JM$. Now, let us prove (c). Let $z \in \ker \sigma_n \subset \mathcal{R}(I, J; M)_n = \oplus_{p+q=n} I^p J^q M u^p v^q \subset M[u, v]$. Thus, $z = a_0 u^n + a_1 u^{n-1} v + \dots + a_{n-1} u v^{n-1} + a_n v^n$, $a_i \in I^{n-i} J^i M$, and $0 = \sigma_n(z) = (a_0 + a_1 + \dots + a_{n-1} + a_n) t^n \in \mathcal{R}(I+J; M)_n = (I+J)^n M t^n$. So $a_0 + a_1 + \dots + a_{n-1} + a_n = 0$. Let us denote:

$$\left\{ \begin{array}{l} b_0 = a_0 \in I^n M \cap JM = I^n JM \\ b_1 = a_0 + a_1 \in I^{n-1} M \cap J^2 M = I^{n-1} J^2 M \text{ and } a_1 = b_1 - b_0 \\ b_2 = a_0 + a_1 + a_2 \in I^{n-2} M \cap J^3 M = I^{n-2} J^3 M \text{ and } a_2 = b_2 - b_1 \\ \dots \\ b_{n-2} = a_0 + \dots + a_{n-2} \in I^2 M \cap J^{n-1} M = I^2 J^{n-1} M \text{ and } a_{n-2} = b_{n-2} - b_{n-3} \\ b_{n-1} = a_0 + \dots + a_{n-1} \in IM \cap J^n M = I J^n M \text{ and } a_{n-1} = b_{n-1} - b_{n-2} \\ a_n = -b_{n-1} \in I J^n M. \end{array} \right.$$

We can rewrite z in $M[u, v]$ in the following manner:

$$\begin{aligned} z &= a_0 u^n + a_1 u^{n-1} v + \dots + a_{n-1} u v^{n-1} + a_n v^n = \\ &= b_0 u^n + (b_1 - b_0) u^{n-1} v + (b_2 - b_1) u^{n-2} v^2 + \dots + \\ &+ (b_{n-2} - b_{n-3}) u^2 v^{n-2} + (b_{n-1} - b_{n-2}) u v^{n-1} + (-b_{n-1}) v^n = \\ &= \underbrace{(b_0 u^{n-1} + b_1 u^{n-2} v + b_2 u^{n-3} v^2 + \dots + b_{n-2} u v^{n-2} + b_{n-1} v^{n-1})}_{p(u, v)} (u - v) := p(u, v)(u - v), \end{aligned}$$

where $p(u, v) \in A[Iu, Jv]_{n-1} \cdot (IJM) = \mathcal{R}(I, J)_{n-1} \cdot (IJM)$. Since by hypothesis $IM \cap JM = IJM$, then $\ker(\sigma_1) = (IJM)(u - v)$, $\ker(\bar{\sigma}_1) = 0$ and $z = p(u, v)(u - v) \in \mathcal{R}(I, J)_{n-1} \cdot (IJM)(u - v) = \mathcal{R}(I, J)_{n-1} \cdot \ker \sigma_1$. Thus $\ker \sigma_n = \mathcal{R}(I, J)_{n-1} \cdot \ker \sigma_1$ for all $n \geq 2$ and $s(E(\sigma)) = 1$. By Lemma 2.1, $s(E(\bar{\sigma}_1)) \leq s(E(\sigma)) = 1$. Therefore $\ker(\bar{\sigma}_n) = \mathcal{G}(I, J)_{n-1} \cdot \ker(\bar{\sigma}_1) = 0$ for all $n \geq 2$ and $\bar{\sigma}$ is an isomorphism. ■

Proposition 2.3 *Let A be a noetherian ring, I, J two ideals of A and M a finitely generated A -module. The following two conditions are equivalent:*

- (i) $\bar{\sigma} : \mathcal{G}(I, J; M) \rightarrow \mathcal{G}(I+J; M)$ is an isomorphism.
- (ii) $I^p M \cap J^q M = I^p J^q M$ for all integers $p, q \geq 1$.

Proof. Remark that we can suppose A is local. By Lemma 2.2, (ii) \Rightarrow (i). Let us see (i) \Rightarrow (ii), proving by double induction in $p, q \geq 1$ that

$$I^p M \cap J^q M \subset I^p(I+J)J^{q-1}M \cap (I+J)^p J^q M.$$

Remark that if $I^p M \cap J^q M \subset I^p(I+J)J^{q-1}M$ for all $p, q \geq 1$, then $I^p M \cap J^q M \subset I^{p+1}M + I^p J^q M$ and $I^p M \cap J^q M \subset I^{p+1}M \cap J^q M + I^p J^q M$. Recursively, and using A is noetherian local and M is finitely generated, $I^p M \cap J^q M \subset (\cap_{r \geq 1} I^{p+r} M \cap J^q M) + I^p J^q M \subset (\cap_{n \geq 1} I^n M) + I^p J^q M = I^p J^q M$, concluding (ii). Take $q = 1$. Let us prove by induction in $p \geq 1$ that

$$I^p M \cap JM \subset I^p(I+J)M \cap (I+J)^p JM.$$

For $p = 1$, we apply Lemma 2.2, (b), using the hypothesis $\overline{\sigma}_1$ is an isomorphism. Suppose

$$I^p M \cap JM \subset I^p(I+J)M \cap (I+J)^p JM$$

is true and let us prove

$$I^{p+1}M \cap JM \subset I^{p+1}(I+J)M \cap (I+J)^{p+1}JM.$$

Then $I^{p+1}M \cap JM \subset I^p M \cap JM \subset (I+J)^p JM$. Consider the short complex of A -modules:

$$I^{p+1}M \cap JM \xrightarrow{\alpha} I^{p+1}M \oplus (I+J)^p JM \xrightarrow{\beta} (I+J)^{p+1}M,$$

where $\alpha(a) = (a, -a)$ and $\beta(a, b) = a + b$. Remark that $\beta \circ \alpha = 0$, β is surjective and that there exists a natural epimorphism γ of A -modules such that $\beta \circ \gamma = \sigma_{p+1}$. If we tensor this short complex by $A/(I+J)$ we obtain:

$$\begin{aligned} (I^{p+1}M \cap JM) \otimes A/(I+J) &\xrightarrow{\overline{\alpha}} I^{p+1}M/I^{p+1}(I+J)M \oplus (I+J)^p JM/(I+J)^{p+1}JM \\ I^{p+1}M/I^{p+1}(I+J)M \oplus (I+J)^p JM/(I+J)^{p+1}JM &\xrightarrow{\overline{\beta}} (I+J)^{p+1}M/(I+J)^{p+2}M, \end{aligned}$$

with $\overline{\beta} \circ \overline{\alpha} = 0$. Since $\overline{\sigma}_{p+1} = \overline{\beta} \circ \overline{\gamma}$ is an isomorphism, then $\overline{\beta}$ is an isomorphism, $\overline{\alpha} = 0$ and

$$I^{p+1}M \cap JM \subset I^{p+1}(I+J)M \cap (I+J)^{p+1}JM.$$

By the symmetry of the problem, the following inclusion is also true for all $q \geq 1$:

$$IM \cap J^q M \subset (I+J)J^q M \cap I(I+J)^q M.$$

In particular, if $I^p M \cap JM \subset I^p(I+J)M$ for all $p \geq 1$, then $I^p M \cap JM \subset I^{p+1}M + I^p JM$ and $I^p M \cap JM \subset I^{p+1}M \cap JM + I^p JM$. Recursively, and using A is noetherian local and M is finitely generated, $I^p M \cap JM \subset (\cap_{r \geq 1} I^{p+r} M \cap JM) + I^p JM \subset (\cap_{n \geq 1} I^n M) + I^p JM = I^p JM$ concluding $I^p M \cap JM = I^p JM$ for all $p \geq 1$. Again, by the symmetry of the problem, $IM \cap J^q M = IJ^q M$ for all $q \geq 1$. Now, suppose

$$I^p M \cap J^q M \subset I^p(I+J)J^{q-1}M \cap (I+J)^p J^q M$$

holds for all $p \geq 1$ and let us prove, by induction in $p \geq 1$, that

$$I^p M \cap J^{q+1}M \subset I^p(I+J)J^q M \cap (I+J)^p J^{q+1}M.$$

Remark that if $I^p M \cap J^q M \subset I^p(I+J)J^{q-1}M$ for all $p \geq 1$, then $I^p M \cap J^q M \subset I^{p+1}M + I^p J^q M$ and $I^p M \cap J^q M \subset I^{p+1}M \cap J^q M + I^p J^q M$. Recursively, and using A is noetherian local and M is finitely generated, $I^p M \cap J^q M \subset (\cap_{r \geq 1} I^{p+r} M \cap J^q M) + I^p J^q M \subset (\cap_{n \geq 1} I^n M) + I^p J^q M = I^p J^q M$ concluding $I^p M \cap J^q M = I^p J^q M$ for all $p \geq 1$. For $p = 1$, we have to show:

$$IM \cap J^{q+1}M \subset I(I+J)J^q M \cap (I+J)J^{q+1}M.$$

We have $IM \cap J^{q+1}M \subset IM \cap J^qM = IJ^qM$. Consider the short complex of A -modules:

$$IM \cap J^{q+1}M \xrightarrow{\alpha} I^{q+1}M \oplus \dots \oplus IJ^qM \oplus J^{q+1}M \xrightarrow{\sigma_{q+1}} (I+J)^{q+1}M,$$

where $\alpha(a) = (0, \dots, 0, a, -a)$. Remark that $\sigma_{q+1} \circ \alpha = 0$. If we tensor this complex by $A/(I+J)$ we obtain $\overline{\sigma}_{q+1} \circ \overline{\alpha} = 0$. Since $\overline{\sigma}_{q+1}$ is an isomorphism, then $\overline{\alpha} = 0$ and

$$IM \cap J^{q+1}M \subset I(I+J)J^qM \cap (I+J)J^{q+1}M.$$

Suppose now true

$$I^pM \cap J^{q+1}M \subset I^p(I+J)J^qM \cap (I+J)^pJ^{q+1}M$$

and let us prove

$$I^{p+1}M \cap J^{q+1}M \subset I^{p+1}(I+J)J^qM \cap (I+J)^{p+1}J^{q+1}M.$$

Then $I^{p+1}M \cap J^{q+1}M \subset I^pM \cap J^{q+1}M \subset (I+J)^pJ^{q+1}M$ and $I^{p+1}M \cap J^{q+1}M \subset I^{p+1}M \cap J^qM = I^{p+1}J^qM$. Consider the short complex of A -modules:

$$I^{p+1}M \cap J^{q+1}M \xrightarrow{\alpha} I^{p+q+1}M \oplus \dots \oplus I^{p+1}J^qM \oplus (I+J)^pJ^{q+1}M \xrightarrow{\beta} (I+J)^{p+q+1}M,$$

where $\alpha(a) = (0, \dots, 0, a, -a)$ and $\beta(a_1, \dots, a_{q+2}) = a_1 + \dots + a_{q+2}$. Remark that $\beta \circ \alpha = 0$, β is surjective and that there exists a natural epimorphism γ of A -modules such that $\beta \circ \gamma = \sigma_{p+q+1}$. If we tensor this complex by $A/(I+J)$ we obtain $\overline{\beta} \circ \overline{\alpha} = 0$. Since $\overline{\sigma}_{p+q+1} = \overline{\beta} \circ \overline{\gamma}$ is an isomorphism, then $\overline{\beta}$ is an isomorphism, $\overline{\alpha} = 0$ and

$$I^{p+1}M \cap J^{q+1}M \subset I^{p+1}(I+J)J^qM \cap (I+J)^{p+1}J^{q+1}JM. \blacksquare$$

Proposition 2.4 *Let A be a commutative ring, I an ideal of A and $\lambda : M \otimes N \rightarrow P$ an epimorphism of A -modules. Consider $f : \mathcal{R}(I; M) \otimes N \rightarrow \mathcal{R}(I; P)$ and $\overline{f} = f \otimes 1_{A/I} : \mathcal{G}(I; M) \otimes N \rightarrow \mathcal{G}(I; P)$ the natural surjective graded morphisms of standard modules. Then, for each integer $n \geq 2$, there exists an exact sequence of A -modules $E(f)_{n+1} \rightarrow E(f)_n \rightarrow E(\overline{f})_n \rightarrow 0$. In particular, if A is noetherian, M, N, P are finitely generated and \overline{f} is an isomorphism, then f is an isomorphism.*

Proof. For each integer $n \geq 1$, the natural morphism $\text{Tor}_1(A/I^n, M) \otimes N \rightarrow \text{Tor}_1(A/I^n, M \otimes N)$ and $\lambda : M \otimes N \rightarrow P$ define the following commutative diagram of exact rows:

$$\begin{array}{ccccccc} \text{Tor}_1(A/I^n, M) \otimes N & \longrightarrow & I^n \otimes M \otimes N & \longrightarrow & I^n M \otimes N & \longrightarrow & 0 \\ \downarrow & & \downarrow 1 \otimes \lambda & & & & \\ 0 & \longrightarrow & \text{Tor}_1(A/I^n, P) & \longrightarrow & I^n \otimes P & \longrightarrow & I^n P \longrightarrow 0 \end{array}$$

We deduce an epimorphism $f_n : I^n M \otimes N \rightarrow I^n P$. On the other hand, $\mathcal{R}(I; M) \otimes M$ is a standard $\mathcal{R}(I)$ -module and $f = \oplus_{n \geq 0} f_n : \mathcal{R}(I; M) \otimes N \rightarrow \mathcal{R}(I; P)$ defines a surjective graded morphism of standard $\mathcal{R}(I)$ -modules. If we tensor f by A/I , we get $\overline{f} : \mathcal{G}(I; M) \otimes N \rightarrow \mathcal{G}(I; P)$ a surjective graded morphism of standard $\mathcal{G}(I)$ -modules.

Let X be an A -module. The following is a commutative diagram of exact columns with rows the last three nonzero terms of the complexes $\mathcal{K}(\mathcal{R}(I; X))_{n+1}$, $\mathcal{K}(\mathcal{R}(I; X))_n$ and $\mathcal{K}(\mathcal{G}(I; X))_n$ (see Proposition 2.6 in [P₂] for more details):

$$\begin{array}{ccccccc}
\mathcal{K}(\mathcal{R}(I; X))_{n+1} & \cdots & \Lambda_2 I \otimes I^{n-1} X & \xrightarrow{\partial_{2,n+1}} & I \otimes I^n X & \xrightarrow{\partial_{1,n+1}} & I^{n+1} X \longrightarrow 0 \\
\downarrow u. & & \downarrow u_2 & & \downarrow u_1 & & \downarrow u_0 \\
\mathcal{K}(\mathcal{R}(I; X))_n & \cdots & \Lambda_2 I \otimes I^{n-2} X & \xrightarrow{\partial_{2,n}} & I \otimes I^{n-1} X & \xrightarrow{\partial_{1,n}} & I^n X \longrightarrow 0 \\
\downarrow v. & & \downarrow v_2 & & \downarrow v_1 & & \downarrow v_0 \\
\mathcal{K}(\mathcal{G}(I; X))_n & \cdots & \Lambda_2 I/I^2 \otimes I^{n-2} X/I^{n-1} X & \xrightarrow{\partial_{2,n}} & I/I^2 \otimes I^{n-1} X/I^n X & \xrightarrow{\partial_{1,n}} & I^n X/I^{n+1} X \longrightarrow 0
\end{array}$$

In other words, $\mathcal{K}(\mathcal{R}(I; X))_{n+1} \xrightarrow{u} \mathcal{K}(\mathcal{R}(I; X))_n \xrightarrow{v} \mathcal{K}(\mathcal{G}(I; X))_n \rightarrow 0$ is an exact sequence of complexes. It induces the morphisms in homology: $H_1(\mathcal{K}(\mathcal{R}(I; X))_{n+1}) \xrightarrow{u} H_1(\mathcal{K}(\mathcal{R}(I; X))_n)$ and $H_1(\mathcal{K}(\mathcal{R}(I; X))_n) \xrightarrow{v} H_1(\mathcal{K}(\mathcal{G}(I; X))_n)$. By Proposition 2.6 in $[\mathbf{P}_2]$, $H_1(\mathcal{K}(\mathcal{R}(I; X))_n) = E(I; X)_n$ and $H_1(\mathcal{K}(\mathcal{G}(I; X))_n) = E(\mathcal{G}(I; X))_n$. Thus we have $E(I; X)_{n+1} \xrightarrow{u} E(I; X)_n \xrightarrow{v} E(\mathcal{G}(I; X))_n$. Since $v \circ u = 0$, then $v \circ u = 0$. Since u_0 is injective, then $\ker v \subset \operatorname{im} u$. Since $H_0(\mathcal{K}(\mathcal{R}(I; X))_{n+1}) = 0$, then v is surjective. So $E(I; X)_{n+1} \xrightarrow{u} E(I; X)_n \xrightarrow{v} E(\mathcal{G}(I; X))_n \rightarrow 0$ is an exact sequence of A -modules. For $X = P$ we get the exact sequence of A -modules: $E(I; P)_{n+1} \xrightarrow{u} E(I; P)_n \xrightarrow{v} E(\mathcal{G}(I; P))_n \rightarrow 0$. Take $X = M$ in $\mathcal{K}(\mathcal{R}(I; X))_{n+1} \xrightarrow{u} \mathcal{K}(\mathcal{R}(I; X))_n \xrightarrow{v} \mathcal{K}(\mathcal{G}(I; X))_n \rightarrow 0$ and tensor it by N . Then we get the exact sequence of complexes

$$\mathcal{K}(\mathcal{R}(I; M))_{n+1} \otimes N \xrightarrow{\alpha = u \otimes 1} \mathcal{K}(\mathcal{R}(I; M))_n \otimes N \xrightarrow{\beta = v \otimes 1} \mathcal{K}(\mathcal{G}(I; M))_n \otimes N \longrightarrow 0.$$

That is, we obtain the exact sequence:

$$\mathcal{K}(\mathcal{R}(I; M) \otimes N)_{n+1} \xrightarrow{\alpha} \mathcal{K}(\mathcal{R}(I; M) \otimes N)_n \xrightarrow{\beta} \mathcal{K}(\mathcal{G}(I; M) \otimes N)_n \longrightarrow 0,$$

which induces the morphisms in homology

$$H_1(\mathcal{K}(\mathcal{R}(I; M) \otimes N)_{n+1}) \xrightarrow{\alpha} H_1(\mathcal{K}(\mathcal{R}(I; M) \otimes N)_n) \xrightarrow{\beta} H_1(\mathcal{K}(\mathcal{G}(I; M) \otimes N)_n).$$

Again, by Proposition 2.6 in $[\mathbf{P}_2]$, $H_1(\mathcal{K}(\mathcal{R}(I; M) \otimes N)_n) = E(\mathcal{R}(I; M) \otimes N)_n$ and $H_1(\mathcal{K}(\mathcal{G}(I; M) \otimes N)_n) = E(\mathcal{G}(I; M) \otimes N)_n$. Moreover, since $\beta \circ \alpha = 0$, then $\beta \circ \alpha = 0$, and since $H_0(\mathcal{K}(\mathcal{R}(I; M) \otimes N)_{n+1}) = 0$, then β is an epimorphism. Thus we have

$$E(\mathcal{R}(I; M) \otimes N)_{n+1} \xrightarrow{\alpha} E(\mathcal{R}(I; M) \otimes N)_n \xrightarrow{\beta} E(\mathcal{G}(I; M) \otimes N)_n \longrightarrow 0$$

with $\beta \circ \alpha = 0$ and β surjective. Remark that since we do not know if $\alpha_0 = u_0 \otimes 1$ is injective, we can not deduce $\ker \beta \subset \operatorname{im} \alpha$. On the other hand, consider $g : \mathbf{S}(I) \otimes M \otimes N \rightarrow \mathcal{R}(I; M) \otimes N$ and $\bar{g} : \mathbf{S}(I/I^2) \otimes M \otimes N \rightarrow \mathcal{G}(I; M) \otimes N$ the natural surjective graded morphisms of standard modules, where $\mathbf{S}(I)$, $\mathbf{S}(I/I^2)$ stands for the symmetric algebras of I and I/I^2 , respectively. By Lemma 2.3 in $[\mathbf{P}_2]$, for each $n \geq 2$, there exists exact sequences of A -modules $E(g)_n \rightarrow E(f \circ g)_n \rightarrow E(f)_n \rightarrow 0$ and $E(\bar{g})_n \rightarrow E(\bar{f} \circ \bar{g})_n \rightarrow E(\bar{f})_n \rightarrow 0$. In other words, we have exact sequences

$$\begin{aligned}
E(\mathcal{R}(I; M) \otimes N)_n &\rightarrow E(\mathcal{R}(I; P))_n \rightarrow E(f)_n \rightarrow 0 \text{ and} \\
E(\mathcal{G}(I; M) \otimes N)_n &\rightarrow E(\mathcal{G}(I; P))_n \rightarrow E(\bar{f})_n \rightarrow 0.
\end{aligned}$$

Consider the following commutative diagram of exact columns:

$$\begin{array}{ccccccc}
E(\mathcal{R}(I; M) \otimes N)_{n+1} & \xrightarrow{\alpha} & E(\mathcal{R}(I; M) \otimes N)_n & \xrightarrow{\beta} & E(\mathcal{G}(I; M) \otimes N)_n & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
E(\mathcal{R}(I; P))_{n+1} & \xrightarrow{u} & E(\mathcal{R}(I; P))_n & \xrightarrow{v} & E(\mathcal{G}(I; P))_n & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
E(f)_{n+1} & & E(f)_n & & E(\bar{f})_n & &
\end{array}$$

The commutativity induces two morphisms $\xi : E(f)_{n+1} \rightarrow E(f)_n$ and $\mu : E(f)_n \rightarrow E(\bar{f})_n$. Since $v \circ u = 0$, then $\mu \circ \xi = 0$. Since v is surjective, then μ is surjective too. Since β is surjective and the middle row is exact, then $\ker \mu \subset \text{im} \xi$. Therefore,

$$E(f)_{n+1} \xrightarrow{\xi} E(f)_n \xrightarrow{\mu} E(\bar{f})_n \longrightarrow 0$$

is an exact sequence of A modules. Finally, if A is noetherian and M, N and P are finitely generated, then $E(f)_n = 0$ for $n \gg 0$ big enough. ■

Theorem 2.5 *Let A be a noetherian ring, I, J two ideals of A and M a finitely generated A -module. The following two conditions are equivalent:*

(i) $\bar{\varphi} : \mathcal{G}(I) \otimes \mathcal{G}(J; M) \rightarrow \mathcal{G}(I + J; M)$ is an isomorphism.

(ii) $\text{Tor}_1(A/I^p, \mathcal{R}(J; M)) = 0$ and $\text{Tor}_1(A/I^p, \mathcal{G}(J; M)) = 0$ for all integers $p \geq 1$.

In particular, $\mathcal{G}(I) \otimes \mathcal{G}(J) \simeq \mathcal{G}(I + J)$ if and only if $\text{Tor}_1(A/I^p, A/J^q) = 0$ and $\text{Tor}_2(A/I^p, A/J^q) = 0$ for all integers $p, q \geq 1$.

Proof. Remark that $\text{Tor}_1(A/I^p, J^q M) = \ker(\pi_{p,q} : I^p \otimes J^q M \rightarrow I^p J^q M)$. Moreover, under the hypothesis $\text{Tor}_1(A/I^p, \mathcal{R}(J; M)) = 0$ for all $p \geq 1$, then the following two conditions are equivalent:

- $\text{Tor}_1(A/I^p, \mathcal{G}(J; M)) = 0$ for all $p \geq 1$.
- $I^p M \cap J^q M = I^p J^q M$ for all $p, q \geq 1$.

Suppose (ii) holds, i.e., $\text{Tor}_1(A/I^p, J^q M) = 0$ and $I^p M \cap J^q M = I^p J^q M$ for all $p, q \geq 1$. Then, $\pi : \mathcal{R}(I) \otimes \mathcal{R}(J; M) \rightarrow \mathcal{R}(I, J; M)$ is an isomorphism and, by Lemma 2.2, $\bar{\sigma} : \mathcal{G}(I, J; M) \rightarrow \mathcal{G}(I + J; M)$ is an isomorphism. Thus $\bar{\varphi} = \bar{\sigma} \circ \bar{\pi}$ is an isomorphism and (i) holds. Let us now prove (i) \Rightarrow (ii). If $\bar{\varphi} = \bar{\sigma} \circ \bar{\pi}$ is an isomorphism, then $\bar{\sigma}$ and $\bar{\pi}$ are two isomorphisms. By Proposition 2.3, $\bar{\sigma}$ an isomorphism implies $I^p M \cap J^q M = I^p J^q M$ for all $p, q \geq 1$. In particular,

$$\begin{aligned}
\mathcal{R}(I; J^q M / J^{q+1} M)_p &= \frac{I^p J^q M + J^{q+1} M}{J^{q+1} M} = \frac{I^p J^q M}{I^p J^q M \cap J^{q+1} M} = \frac{I^p J^q M}{I^p J^{q+1} M} = \mathcal{G}(J; I^p M)_q \text{ and} \\
\mathcal{G}(I; J^q M / J^{q+1} M)_p &= \frac{I^p J^q M + J^{q+1} M}{I^{p+1} J^q M + J^{q+1} M} = \frac{I^p J^q M}{(I + J) I^p J^q M} = \mathcal{G}(I, J; M)_{p,q}.
\end{aligned}$$

Fix $q \geq 1$. Since $\bar{\pi}_{p,q} : \mathcal{G}(I)_p \otimes \mathcal{G}(J; M)_q \rightarrow \mathcal{G}(I, J; M)_{p,q}$ is an isomorphism for all $p \geq 1$ and $\mathcal{G}(I, J; M)_{p,q} = \mathcal{G}(I; J^q M / J^{q+1} M)_p$, then $\bar{\pi}_{*,q} : \mathcal{G}(I) \otimes J^q M / J^{q+1} M \rightarrow \mathcal{G}(I; J^q M / J^{q+1} M)$ is an isomorphism for all $q \geq 1$. By Proposition 2.4, we have $\mathcal{R}(I) \otimes J^q M / J^{q+1} M \rightarrow \mathcal{R}(I; J^q M / J^{q+1} M)$ is an isomorphism for all $q \geq 1$. In other words, $I^p \otimes \mathcal{G}(J; M) \rightarrow \mathcal{G}(J; I^p M)$ is an isomorphism for all $p \geq 1$ (since $\mathcal{R}(I; J^q M / J^{q+1} M)_p = \mathcal{G}(J; I^p M)_q$). By Proposition 2.4, $I^p \otimes \mathcal{R}(J; M) \rightarrow \mathcal{R}(J; I^p M)$ is an isomorphism for all $p \geq 1$. So $\pi : \mathcal{R}(I) \otimes \mathcal{R}(J; M) \rightarrow \mathcal{R}(I, J; M)$ is an isomorphism and $\text{Tor}_1(A/I^p, \mathcal{R}(J; M)) = 0$ for all $p \geq 1$. ■

3 Some examples

Example 3.1 Let A be a noetherian local ring, I, J two ideals of A and M a finitely generated A -module. If $I = (x)$ is principal and x A -regular, then $\overline{\varphi} : \mathcal{G}(I) \otimes \mathcal{G}(J; M) \rightarrow \mathcal{G}(I + J; M)$ is an isomorphism if and only if x is a nonzero divisor in $\mathcal{R}(J; M)$ and in $\mathcal{G}(J; M)$. Indeed, let $\mathcal{K}(y; N)$ denote the Koszul complex of a sequence of elements $y = y_1, \dots, y_m$ of A with respect to an A -module N and let $H_i(y; N)$ denote its i -th Koszul homology group. Then $\text{Tor}_1(A/I, N) = H_1(\mathcal{K}(x; A) \otimes N) = H_1(x; M) = 0$ if and only if x is a non-zero-divisor in N .

Example 3.2 Let A be a noetherian local ring and let $I = (x)$ and $J = (y)$ be two principal ideals of A . If $(0 : x) \subset (y)$ and $(0 : y) \subset (x)$, then $\overline{\varphi} : \mathcal{G}(I) \otimes \mathcal{G}(J) \rightarrow \mathcal{G}(I + J)$ is an isomorphism if and only if x, y is an A -regular sequence.

Example 3.3 Let R be a noetherian local ring and let z, t be an R -regular sequence. Let $A = R/(zt)$, $x = z + (zt)$, $y = t + (zt)$, $I = (x)$ and $J = (y)$. Then $\overline{\sigma} : \mathcal{G}(I, J) \rightarrow \mathcal{G}(I + J)$ is an isomorphism, but $\overline{\pi} : \mathcal{G}(I) \otimes \mathcal{G}(J) \rightarrow \mathcal{G}(I, J)$ is not an isomorphism.

An example of a pair of ideals I, J with the property $\text{Tor}_1(A/I^p, A/J^q) = 0$ for all integers $p, q \geq 1$ arises from a product of affine varieties (see [V], pages 130 to 136, and specially Proposition 5.5.7). The next result is well known (see, for instance, [HIO]). We give here a proof for the sake of completeness.

Proposition 3.4 Let A be a noetherian local ring, I and J two ideals of A and M a finitely generated A -module. Let $x = x_1, \dots, x_r$ be a system of generators of I and $y = y_1, \dots, y_r$, $y_i = \overline{x}_i = x_i + J$, a system of generators of the ideal $\overline{I} = I + J/J$ of the quotient ring $\overline{A} = A/J$. If $\mathcal{G}(J)$ and $\mathcal{G}(J; M)$ are free \overline{A} -modules and y is an \overline{A} -regular sequence in \overline{I} , then x is an A -regular sequence in I and then $\overline{\varphi} : \mathcal{G}(I) \otimes \mathcal{G}(J; M) \rightarrow \mathcal{G}(I + J; M)$ is an isomorphism.

Proof. Since, for all $q \geq 1$, $J^q M / J^{q+1} M$ is \overline{A} -free and y is an \overline{A} -regular sequence, then

$$0 = \text{Tor}_1^{\overline{A}}(\overline{A}/\overline{I}, J^q M / J^{q+1} M) = H_1(\mathcal{K}(y; \overline{A}) \otimes J^q M / J^{q+1} M) = H_1(y; J^q M / J^{q+1} M).$$

So y is a $J^q M / J^{q+1} M$ -regular sequence in \overline{I} for all $q \geq 1$. In particular, x is a $J^q M / J^{q+1} M$ -regular sequence in I and $H_1(x; J^q M / J^{q+1} M) = 0$ for all $q \geq 1$. Using the long exact sequences in homology associated to the short exact sequences of A -modules $0 \rightarrow J^q M / J^{q+1} M \rightarrow M / J^{q+1} M \rightarrow M / J^q M \rightarrow 0$, we deduce $H_1(x; M / J^q M) = 0$ and x is an $M / J^q M$ -regular sequence in I for all $q \geq 1$. In particular, x is an M -regular sequence in I . Analogously, but using the hypothesis $\mathcal{G}(J)$ is \overline{A} -free, we deduce x is an A -regular sequence in I . Therefore

$$\begin{aligned} \text{Tor}_i(A/I, M) &= H_i(\mathcal{K}(x; A) \otimes M) = H_i(\mathcal{K}(x; M)) = 0 \text{ and} \\ \text{Tor}_i(A/I, M / J^q M) &= H_i(\mathcal{K}(x; A) \otimes M / J^q M) = H_i(\mathcal{K}(x; M / J^q M)) = 0. \end{aligned}$$

Using the long exact sequences in homology associated to the short exact sequences

$$0 \rightarrow J^q M \rightarrow M \rightarrow M / J^q M \rightarrow 0 \text{ and } 0 \rightarrow J^q M / J^{q+1} M \rightarrow M / J^{q+1} M \rightarrow M / J^q M \rightarrow 0,$$

we deduce $\text{Tor}_1(A/I, \mathcal{R}(J; M)) = 0$ and $\text{Tor}_1(A/I, \mathcal{G}(J; M)) = 0$. Since I^p / I^{p+1} is A/I -free, then $\text{Tor}_1(I^p / I^{p+1}, \mathcal{R}(J; M)) = \text{Tor}_1(A/I, \mathcal{R}(J; M)) \otimes I^p / I^{p+1} = 0$ and $\text{Tor}_1(I^p / I^{p+1}, \mathcal{G}(J; M)) = \text{Tor}_1(A/I, \mathcal{G}(J; M)) \otimes I^p / I^{p+1} = 0$. Applying the long exact sequences in homology to the short exact sequences $0 \rightarrow I^p / I^{p+1} \rightarrow A / I^{p+1} \rightarrow A / I^p \rightarrow 0$, we deduce $\text{Tor}_1(A / I^p, \mathcal{R}(J; M)) = 0$ and $\text{Tor}_1(A / I^p, \mathcal{G}(J; M)) = 0$ for all $p \geq 1$. ■

4 Relation type of tensor products

Lemma 4.1 *Let U be a standard A -algebra and F a standard U -module. If M is an A -module, then $F \otimes M$ is a standard U -module and $\text{rt}(F \otimes M) \leq \text{rt}(F)$. If $\lambda : M \rightarrow N$ is an epimorphism of A -modules, then $1 \otimes \lambda : F \otimes M \rightarrow F \otimes N$ is a surjective graded morphism of standard U -modules. Moreover, for each integer $n \geq 1$, $\ker(1_{F_n} \otimes \lambda) = U_1 \cdot \ker(1_{F_{n-1}} \otimes \lambda)$. In particular, for each $n \geq 1$, there exists an epimorphism of A -modules $E(F \otimes M)_n \rightarrow E(F \otimes N)_n$ and $\text{rt}(F \otimes N) \leq \text{rt}(F \otimes M)$.*

Proof. Clearly $F \otimes M$ is a standard U -module and $1 \otimes \lambda : F \otimes M \rightarrow F \otimes N$ is a surjective graded morphism of standard U -modules. By Proposition 2.6 in $[\mathbf{P}_2]$, for each $n \geq \text{rt}(F) + 1$, the following sequence is exact:

$$\Lambda_2(U_1) \otimes F_{n-2} \rightarrow U_1 \otimes F_{n-1} \rightarrow F_n \rightarrow 0.$$

If we tensor it by M , we obtain the exact sequence

$$\Lambda_2(U_1) \otimes F_{n-2} \otimes M \rightarrow U_1 \otimes F_{n-1} \otimes M \rightarrow F_n \otimes M \rightarrow 0,$$

for all $n \geq \text{rt}(F) + 1$. Thus $E(F \otimes M)_n = 0$ for all $n \geq \text{rt}(F) + 1$ and $\text{rt}(F \otimes M) \leq \text{rt}(F)$. Consider the following commutative diagram of exact columns and rows:

$$\begin{array}{ccccccc}
 & & & & (\ker \partial_n) \otimes M & \xrightarrow{1 \otimes \lambda} & (\ker \partial_n) \otimes N \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & U_1 \otimes F_{n-1} \otimes M & \xrightarrow{1 \otimes 1 \otimes \lambda} & U_1 \otimes F_{n-1} \otimes N \longrightarrow 0 \\
 & & & & \downarrow \partial_n \otimes 1_M & & \downarrow \partial_n \otimes 1_N \\
 U_1 \otimes \ker(1_{F_{n-1}} \otimes \lambda) & \longrightarrow & & & F_n \otimes M & \longrightarrow & F_n \otimes N \longrightarrow 0 \\
 \downarrow & & & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \ker(1_{F_n} \otimes \lambda) & \longrightarrow & & &
 \end{array}$$

Using a diagram chasing argument, one deduces $\ker(1_{F_n} \otimes \lambda) = U_1 \cdot \ker(1_{F_{n-1}} \otimes \lambda)$ for all $n \geq 1$. If $g : X \rightarrow F \otimes M$ is a symmetric presentation of $F \otimes M$, then, by Lemma 2.3 in $[\mathbf{P}_2]$, there exists an exact sequence of A -modules $E(g)_n \rightarrow E((1 \otimes \lambda) \circ g)_n \rightarrow E(1 \otimes \lambda)_n \rightarrow 0$ for all $n \geq 1$. But $E(g)_n = E(F \otimes M)_n$, $E((1 \otimes \lambda) \circ g)_n = E(F \otimes N)_n$ and $E(1 \otimes \lambda)_n = 0$ for all $n \geq 1$. Thus $E(F \otimes M)_n \rightarrow E(F \otimes N)_n$ is surjective for all $n \geq 1$ and $\text{rt}(F \otimes N) \leq \text{rt}(F \otimes M)$. ■

Theorem 4.2 *Let A be a commutative ring, U and V two standard A -algebras and F a standard U -module and G a standard V -module. Then $U \otimes V$ is a standard A -algebra, $F \otimes G$ is a standard $U \otimes V$ -module and $\text{rt}(F \otimes G) \leq \max(\text{rt}(F), \text{rt}(G))$.*

Proof. Clearly $U \otimes V$ is a standard A -algebra and $F \otimes G$ is a standard $U \otimes V$ -module. Take $\varphi : X \rightarrow F$ and $\psi : Y \rightarrow G$ two symmetric presentations of F and G , respectively. Then $\varphi \otimes \psi : X \otimes Y \rightarrow F \otimes G$ is a symmetric presentation of $F \otimes G$. Since $\varphi \otimes \psi = (\varphi \otimes 1_G) \circ (1_X \otimes \psi)$, then, for each integer $n \geq 2$, there exists an exact sequence of A -modules

$$E(1_X \otimes \psi)_n \rightarrow E(\varphi \otimes \psi)_n \rightarrow E(\varphi \otimes 1_G)_n \rightarrow 0.$$

Since $\psi : Y \rightarrow G$ is a symmetric presentation of G , then $1_{X_0} \otimes \psi : X_0 \otimes Y \rightarrow X_0 \otimes G$ is a symmetric presentation of $X_0 \otimes G$ and $E(X_0 \otimes G)_n = E(1_{X_0} \otimes \psi)_n$. Using Lemma 4.1, $\ker(1_{X_i} \otimes \psi_{n-i}) =$

$U_1 \cdot \ker(1_{X_{i-1}} \otimes \psi_{n-i})$ for all $i \geq 1$. Then

$$\begin{aligned} E(1_X \otimes \psi)_n &= \frac{\ker(1_X \otimes \psi)_n}{(U \otimes V)_1 \cdot \ker(1_X \otimes \psi)_{n-1}} = \\ &= \frac{\oplus_{i=0}^n \ker(1_{X_i} \otimes \psi_{n-i})}{\left(\oplus_{i=0}^{n-1} U_1 \cdot \ker(1_{X_i} \otimes \psi_{n-i})\right) + \left(\oplus_{i=0}^{n-1} V_1 \cdot \ker(1_{X_i} \otimes \psi_{n-i})\right)} = \\ &= \frac{\ker(1_{X_0} \otimes \psi_n)}{V_1 \cdot \ker(1_{X_0} \otimes \psi_{n-1})} \oplus \frac{\ker(1_{X_1} \otimes \psi_{n-1})}{U_1 \cdot \ker(1_{X_0} \otimes \psi_{n-1}) + V_1 \cdot \ker(1_{X_1} \otimes \psi_{n-2})} \oplus \dots \oplus \\ &= \frac{\ker(1_{X_{n-1}} \otimes \psi_1)}{U_1 \cdot \ker(1_{X_{n-2}} \otimes \psi_1) + V_1 \cdot \ker(1_{X_{n-1}} \otimes \psi_0)} \oplus \frac{\ker(1_{X_n} \otimes \psi_0)}{U_1 \cdot \ker(1_{X_{n-1}} \otimes \psi_0)} = E(1_{X_0} \otimes \psi)_n. \end{aligned}$$

Therefore $E(1_X \otimes \psi)_n = E(1_{X_0} \otimes \psi)_n = E(X_0 \otimes G)_n$ for all $n \geq 1$. Analogously, $E(\varphi \otimes 1_G)_n = E(\varphi \otimes 1_{G_0})_n = E(F \otimes G_0)_n$ for all $n \geq 1$. Hence there exists an exact sequence of A -modules

$$E(X_0 \otimes G)_n \rightarrow E(F \otimes G)_n \rightarrow E(F \otimes G_0)_n \rightarrow 0$$

for all $n \geq 2$ and, by Lemma 4.1, $\text{rt}(F \otimes G) \leq \max(\text{rt}(F \otimes G_0), \text{rt}(X_0 \otimes G)) \leq \max(\text{rt}(F), \text{rt}(G))$. ■

Remark 4.3 Let A be a commutative ring and let U and V be two standard A -algebras. If $\text{Tor}_1^A(U, V) = 0$, then $E(U \otimes V) = E(U) \oplus E(V)$. This follows from the characterization $E(U) = H_1(A, U, A)$ (see Remark 2.3 in [P₁]) and Proposition 19.3 in [A].

5 Uniform bounds

Lemma 5.1 Let (A, \mathfrak{m}) be a noetherian local ring and M be a finitely generated A -module. Let \mathfrak{p} a prime ideal of A such that A/\mathfrak{p} is regular local and $\mathcal{G}(\mathfrak{p})$ and $\mathcal{G}(\mathfrak{p}; M)$ are free A/\mathfrak{p} -modules. Then $\text{rt}(\mathfrak{m}; M) \leq \text{rt}(\mathfrak{p}; M)$.

Proof. Since A/\mathfrak{p} is regular local, there exists a sequence of elements $x = x_1, \dots, x_r$ in A such that $y = y_1, \dots, y_r$, defined by $y_i = x_i + \mathfrak{p}$, is a system of generators of $\mathfrak{m}/\mathfrak{p}$ and an \overline{A} -regular sequence. Let I be the ideal of A generated by x . In particular, $I + \mathfrak{p}/\mathfrak{p} = \mathfrak{m}/\mathfrak{p}$ and $I + \mathfrak{p} = \mathfrak{m}$. By Proposition 3.4, x is an A -regular sequence and $\text{Tor}_1(A/I^p, \mathcal{R}(\mathfrak{p}; M)) = 0$ and $\text{Tor}_1(A/I^p, \mathcal{G}(\mathfrak{p}; M)) = 0$ for all $p \geq 1$. By Theorem 2.5, $\overline{\varphi} : \mathcal{G}(I) \otimes \mathcal{G}(\mathfrak{p}; M) \rightarrow \mathcal{G}(\mathfrak{m}; M)$ is an isomorphism. By Theorem 4.2, $\text{rt}(\mathcal{G}(\mathfrak{m}; M)) \leq \max(\text{rt}(\mathcal{G}(I)), \text{rt}(\mathcal{G}(\mathfrak{p}; M)))$. By Remark 2.7 in [P₂], $\text{rt}(\mathcal{G}(J; M)) = \text{rt}(J; M)$ for any ideal J of A . Since I is generated by a regular sequence, then $\text{rt}(I) = 1$ (see, for instance, [V] page 30). Thus $\text{rt}(\mathfrak{m}; M) \leq \text{rt}(\mathfrak{p}; M)$. ■

Next result is a slight generalization of a well known Theorem of Duncan and O'Carroll [DO]. In fact the proof of our theorem is directly inspired in their. We sketch it here for the sake of completeness.

Theorem 5.2 Let A be an excellent (or $J - 2$) ring and let M be a finitely generated A -module. Then there exists an integer $s \geq 1$ such that, for all maximal ideals \mathfrak{m} of A , the relation type of \mathfrak{m} with respect to M satisfies $\text{rt}(\mathfrak{m}; M) \leq s$.

Proof. For every $\mathfrak{p} \in \text{Spec}(A)$, let us construct a non-empty open subset $U(\mathfrak{p})$ of $V(\mathfrak{p}) = \{\mathfrak{q} \in \text{Spec}(A) \mid \mathfrak{q} \supseteq \mathfrak{p}\} \simeq \text{Spec}(A/\mathfrak{p})$. Remark that A/\mathfrak{p} is a noetherian domain, $\mathcal{G}(\mathfrak{p})$ is a finitely generated A/\mathfrak{p} -algebra and $\mathcal{G}(\mathfrak{p}; M)$ is a finitely generated $\mathcal{G}(\mathfrak{p})$ -module. By Generic Flatness (Theorem 22.A in [M]), there exist $f, g \in A - \mathfrak{p}$ such that $\mathcal{G}(\mathfrak{p})_f$ is an $(A/\mathfrak{p})_f$ -free module and $\mathcal{G}(\mathfrak{p}; M)_g$ is an $(A/\mathfrak{p})_g$ -free

module. Since A is $J - 2$, the set $\text{Reg}(A/\mathfrak{p}) = \{\mathfrak{q} \in V(\mathfrak{p}) \mid (A/\mathfrak{p})_{\mathfrak{q}} \text{ is regular local}\}$ is a non-empty open subset of $V(\mathfrak{p})$. Define $U(\mathfrak{p})$ as the intersection $D(f) \cap D(g) \cap \text{Reg}(A/\mathfrak{p}) = \{\mathfrak{q} \in V(\mathfrak{p}) \mid \mathfrak{q} \not\supseteq f, \mathfrak{q} \not\supseteq g, (A/\mathfrak{p})_{\mathfrak{q}} \text{ is regular local}\}$, which is a non-empty open subset of $V(\mathfrak{p})$. Remark that for all $\mathfrak{q} \in U(\mathfrak{p})$, $(A/\mathfrak{p})_{\mathfrak{q}}$ is regular local and $\mathcal{G}(\mathfrak{p})_{\mathfrak{q}}$ and $\mathcal{G}(\mathfrak{p}; M)_{\mathfrak{q}}$ are free $\mathcal{G}(\mathfrak{p})_{\mathfrak{q}}$ -modules. By Lemma 5.1, $\text{rt}(\mathfrak{q}A_{\mathfrak{q}}; M_{\mathfrak{q}}) \leq \text{rt}(\mathfrak{p}A_{\mathfrak{q}}; M_{\mathfrak{q}}) \leq \text{rt}(\mathfrak{p}; M)$ for all $\mathfrak{q} \in U(\mathfrak{p})$. In particular, $\text{rt}(\mathfrak{m}; M) \leq \text{rt}(\mathfrak{p}; M)$ for all maximal ideals $\mathfrak{m} \in U(\mathfrak{p})$. For each minimal prime \mathfrak{p}_i of A , let $V(\mathfrak{p}_i) - U(\mathfrak{p}_i) = V(\mathfrak{p}_{i,1}) \cup \dots \cup V(\mathfrak{p}_{i,r_i})$ be the decomposition into irreducible closed subsets of the proper closed subset $V(\mathfrak{p}_i) - U(\mathfrak{p}_i)$, $\mathfrak{p}_{i,j} \in \text{Spec}(A)$, $\mathfrak{p}_{i,j} \not\supseteq \mathfrak{p}_i$. Since A is noetherian, $\text{Spec}(A)$ can be covered by finitely many locally closed sets of type $U(\mathfrak{p})$, i.e., there exists a finite number of prime ideals $\mathfrak{q}_1, \dots, \mathfrak{q}_m$, such that $\text{Spec}(A) = \cup_{i=1}^m U(\mathfrak{q}_i)$. Hence, $\text{rt}(\mathfrak{m}; M) \leq \max\{\text{rt}(\mathfrak{q}_i; M) \mid i = 1, \dots, m\}$ for any maximal ideal \mathfrak{m} of A . ■

Using Theorem 2 in [P₂] we deduce the result of Duncan and O'Carroll in [DO].

Corollary 5.3 [DO] *Let A be an excellent (or $J - 2$) ring and let $N \subseteq M$ be two finitely generated A -modules. Then there exists an integer $s \geq 1$ such that, for all integers $n \geq s$ and for all maximal ideals \mathfrak{m} of A , $\mathfrak{m}^n M \cap N = \mathfrak{m}^{n-s}(\mathfrak{m}^s M \cap N)$.*

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